Sheaves of modules

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start of 06.11.24 lec

$(X, \mathcal{O}_X)$ be a ringed space.

Notation. $\mathrm{Mod}_{\mathcal{O}_X}$ = category of sheaves of $\mathcal{O}_X$-mods on X. =: category of $\mathcal{O}_X$-mods.

objects of $\mathrm{Mod}_{\mathcal{O}_X}$ = sheaves of $\mathcal{O}_X$-mods on X

A morphism $f: \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-mods is a map of sheaves s.t. $f_{|U}: \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$ lin for $U \underset{\text{open}}{\subseteq} X$. Such a morphism is called $\mathcal{O}_X$-linear.

e.g.: $g: Y \to X$ morphism of schemes
$g_*\mathcal{O}_Y \in \mathrm{Mod}_{\mathcal{O}_X}$, $g^\#: \mathcal{O}_X \to g_*\mathcal{O}_Y$ is $\mathcal{O}_X$-lin by definition.

§ Operation on $\mathrm{Mod}_{\mathcal{O}_X}$.

- Given $\mathcal{O}_X$-linear $f: \mathcal{F} \to \mathcal{G}$, $\ker f$, $\operatorname{coker} f$, $\operatorname{im}(f)$ are in $\mathrm{Mod}_{\mathcal{O}_X}$.
- $\mathrm{Mod}_{\mathcal{O}_X}$ admits $\otimes_{\mathcal{O}_X}$ pdt.
  $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) :=$ sheafification of the presheaf
  $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$
- Admits an internal hom sheaf
  $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$
- $\mathrm{Mod}_{\mathcal{O}_X}$ admits arbitrary direct sum and product

Ex. Check $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

- i) $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map of ringed spaces. For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_Y}$, $f_*\mathcal{F}$ is a $f_*\mathcal{O}_Y$ module; naturally an $\mathcal{O}_X$-mod: $f_*\mathcal{F}$ is an $\mathcal{O}_X$-mod via the map
  Then $f_*\mathcal{O}_Y$
  $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$.

Def. $(f, f^\#): (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ map of ringed spaces, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. The $f^*\mathcal{G}$ = pull back of $\mathcal{G}$ via f is $f^{-1}\mathcal{G} \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_Y$
[$\mathcal{O}_X \to f_*\mathcal{O}_Y$, gives $f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$, the $\otimes$ is taken on the ringed space $(Y, f^{-1}\mathcal{O}_X)$]

Rmk. 1) $f^*\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_Y}$ (Ex).

2) $(Y, \mathcal{O}_Y)$, $(X, \mathcal{O}_X)$ locally ringed spaces. (eg schemes).
$(f^*\mathcal{G})_y = \mathcal{G}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}$

Recall. $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$ induces $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$.

§ A construction.

Let A be a ring, $M \in \mathrm{Mod}_A$. Consider the presheaf $\tilde{M}$ on $\operatorname{Spec}(A)$.

$\tilde{M}(U) = \{$ set maps $s: U \to \coprod_{p \in U} M_p \mid$ (i) $s(p) \in M_p$ (ii) for $p \in U$, $\exists\, p \in V \underset{\text{open}}{\subseteq} U$ and $m \in M$ s.t. $s(q) = m/f \in M_q$ for some $f \notin \bigcup_{q \in V} q \}$

Thm (Prop 5.1, Hart)
a) $\tilde{M}$ is an $\mathcal{O}_X$-mod.
b) for $p \in \operatorname{Spec}(A)$, $(\tilde{M})_p \xrightarrow{\sim} M_p$ as A_p modules
c) For $f \in A$, the natural map $M_f \xrightarrow{\sim} \tilde{M}(D(f))$ is an isom.
d) In particular $\Gamma(\operatorname{Spec}(A), \tilde{M}) \leftarrow M$ is an isom.

Pf. (a), (b) clear, note (c) $\Rightarrow$ (d)

(c) **Lemma:** Let $s \in \Gamma(\tilde{M}, \operatorname{Spec}(A))$ such that $s_{|D(g)} = 0$ for some $g \in A$. Then $g^m \cdot s = 0$ for some $m \in \mathbb{N}_{\geq 0}$.

Pf. Choose $g_1, g_2, \dots g_r$ such that $s = m_i/g_i$ on $D(g_i)$ and $\bigcup_{i=1}^r D(g_i) = \operatorname{Spec}(A)$.
Since $s_{|D(g_i) \cap D(g)} = s_{|D(g_i g)} = 0$
$m_i/g_i = 0 \in M_q$ $q \in D(g_i g)$
$\Rightarrow m_i/g_i = 0 \in M_{g_i g} = M_{g_i}[1/g]$
$\Rightarrow g^{t_i} \cdot \frac{m_i}{g_i} = 0$ in A_{g_i} for some t_i
$\Rightarrow g^{t_i} \cdot s_{|D(g_i)} = 0$
Take $m = \max\{t_1, t_2 \dots t_r\}$.
Then $g^m \cdot s_{|D(g_i)} = 0$ $\forall i \Rightarrow g^m \cdot s = 0$ ∎

Thm. c Given $s \in \tilde{M}(D(f))$, choose $g_1, g_2, \dots, g_r$ such that
- $D(f) = \bigcup_{i=1}^r D(g_i)$ and
- On $D(g_i)$, $s = m_i/g_i$ for some $m_i \in M$.

i.e. $g_i s = \frac{m_i}{1}$ on $D(g_i)$
By Lemma $g_i^{n_i+1} s = g_i^{n_i} \cdot m_i \in \Gamma(\operatorname{Spec} A[1/f], \tilde{M})$ for some n_i (∗)
$\bigcup_{i=1}^r D(g_i) = \bigcup_{i=1}^r D(g_i^{n_i+1}) = D(f)$
$\Rightarrow V(f) = V(g_1^{n_1+1}, g_2^{n_2+1} \dots, g_r^{n_r+1})$
$\Rightarrow f \in \sqrt{(g_1^{n_1+1} \dots g_r^{n_r+1})}$
$\Rightarrow f^m = g_1^{n_1+1}\beta_1 + \dots + g_r^{n_r+1}\beta_r$ for some m & $\beta_1 \dots \beta_r \in A$.
in $\Gamma(D(f), \tilde{M})$
(∗) $\Rightarrow f^m s = (\sum g_i^{n_i+1}\beta_i) s = \sum g_i^{n_i} \beta_i \cdot m_i \in M$
$\Rightarrow s \in \operatorname{Im}(M_f \to \tilde{M}(D(f)))$.

Rmk. Given $M \in \mathrm{Mod}_A$, consider the presheaf $\mathcal{M}$ of $\mathcal{O}_X$ modules where $X = \operatorname{Spec} A$:
$\mathcal{M}(V) = \{ m/f \mid f \notin \bigcup q,\ q \in V \}$.
Then $\tilde{M}$ is the sheafification of $\mathcal{M}$.

Thm. Given an A-mod map $f: M \to N$, one naturally gets an $\mathcal{O}_{\operatorname{Spec} A}$ linear map $\tilde{f}: \tilde{M} \to \tilde{N}$. Moreover $\widetilde{\mathrm{id}_M} = \mathrm{id}_{\tilde{M}}$ and $\widetilde{f \cdot g} = \tilde{f} \cdot \tilde{g}$.
So $M \mapsto \tilde{M}$ is a functor from the category of A-modules to the category of $\mathcal{O}_{\operatorname{Spec}(A)}$-modules.

Thm. The functor $M \mapsto \tilde{M}$ above has the following properties.
i) The natural map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathcal{O}_{\operatorname{Spec}(A)}}(\tilde{M}, \tilde{N})$ is an isom.
(ii) $X = \operatorname{Spec} A$, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \to \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F}))$ is an isom. Conversely for $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$, if the natural map $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F}) \xrightarrow{\psi^{\mathcal{G}}_{\mathcal{F}}} \mathrm{Hom}_A(\Gamma(X, \mathcal{G}), \Gamma(X, \mathcal{F}))$ is an isomorphism, then $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ for some $M \in \mathrm{Mod}_A$.
(iii) $(\widetilde{\bigoplus_{i \in I} M_i}) = \bigoplus_{i \in I} \tilde{M}_i$ in $\mathrm{Mod}_{\mathcal{O}_{\operatorname{Spec}(A)}}$.
(iv) $\widetilde{M \otimes_A N} \xrightarrow{\sim} \tilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec}(A)}} \tilde{N}$.
(v) $\varphi: A \to B$ ring homo, $M \in \mathrm{Mod}_A$, $(\varphi^\#)^*(\tilde{M}) \xrightarrow{\sim} \widetilde{M \otimes_A B}$ in $\mathrm{Mod}_{\operatorname{Spec} B}$
(vi) φ as in (v), $N \in \mathrm{Mod}_B$, $(\varphi^\#)_*\tilde{N} = \tilde{N}$
where on the right N is considered as an A-mod via restriction of scalars.

Pf (ii) sufficient condn ensuring $\mathcal{G} \xrightarrow{\sim} \tilde{M}$.
Take $M = \Gamma(X, \mathcal{G})$. Our assumption gives a map $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \tilde{M}) \xrightarrow[\sim]{\psi^{\mathcal{G}}_{\tilde{M}}} \mathrm{Hom}_A(\Gamma(X, \mathcal{G}), M)$ gives a map $\varphi_1: \mathcal{G} \to \tilde{M}$ such that $\psi^{\mathcal{G}}_{\tilde{M}}(\varphi_1) = \mathrm{id}_M$.
The isom $\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \xrightarrow[\sim]{\psi^{\tilde{M}}_{\mathcal{G}}} \mathrm{Hom}_A(M, M)$ gives $\varphi_2: \tilde{M} \to \mathcal{G}$ such that $\psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = \mathrm{id}_M$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) \arrow[r, "\psi^{\tilde{M}}_{\mathcal{G}}"] \arrow[d, "\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}{,}\varphi_1)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_A(M, M) \arrow[d, "\mathrm{Hom}(M{,}\mathrm{id}) = \mathrm{id}"] \\
\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{M}) \arrow[r, "\psi^{\tilde{M}}_{\tilde{M}}"'] & \mathrm{Hom}(M, M)
\end{tikzcd}$$

So $\psi^{\tilde{M}}_{\mathcal{G}}(\varphi_2) = \psi^{\tilde{M}}_{\tilde{M}}(\varphi_1 \cdot \varphi_2) = \mathrm{id}$
Since $\psi^{\tilde{M}}_{\tilde{M}}$ is an isom $\varphi_1 \cdot \varphi_2 = \mathrm{id}$.

Thm. $M, N \in \mathrm{Mod}_A$. $f: M \to N$ A-linear.
Then $\ker \tilde{f} \cong \widetilde{\ker f}$, $\widetilde{\operatorname{coker} f} \cong \operatorname{coker} \tilde{f}$

Pf. Have $\ker f \to M \to N$. This gives a complex $\widetilde{\ker f} \to \tilde{M} \to \tilde{N}$. This gives a map $\widetilde{\ker f} \to \ker \tilde{f}$, which is an isom at every stalk.

Thm. $0 \to M' \to M \to M'' \to 0$ is exact in Mod_A
$\Leftrightarrow$ $0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$ is exact in $\mathrm{Mod}_{\mathcal{O}_X}$.

§ Quasi coherent (QCoh) sheaves (ref stacks project tag 01BD)

$(X, \mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$

Def. $\mathcal{F}$ is QCoh if for every $x \in X$, $\exists$ a nbhd U of x and an exact seq
$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}_{|U} \to 0$

Thm. $X = \operatorname{Spec}(A)$, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is QCoh iff $\mathcal{G} \xrightarrow{\sim} \tilde{M}$ in $\mathrm{Mod}_{\mathcal{O}_X}$.

Pf. $\Leftarrow$ Choose a presentation $\bigoplus_{j \in J} A \to \bigoplus_{i \in I} A \to M \to 0$
This gives an exact seq $\bigoplus_{j \in J} \mathcal{O}_X \to \bigoplus_{i \in I} \mathcal{O}_X \to \tilde{M} \to 0$.

$\Rightarrow$ Choose $f_1, f_2, \dots f_r \in A$ such that $X = \bigcup_{j=1}^r D(f_j)$ and for each j, have an exact seq
$\bigoplus_J \mathcal{O}_{D(f_j)} \to \bigoplus_I \mathcal{O}_{D(f_j)} \to \mathcal{G}_{|D(f_j)} \to 0$
Since $\mathcal{O}_{D(f_j)} = \widetilde{A[1/f_j]}$, they are QCoh.
So the coker $\mathcal{G}_{|D(f_j)} \xrightarrow{\sim} \widetilde{\Gamma(D(f_j), \mathcal{G})}$
Let $i_j: D(f_j) \hookrightarrow X$, $i_{j_1 j_2}: D(f_{j_1}) \cap D(f_{j_2}) \hookrightarrow X$ ($D(f_{j_1} f_{j_2})$)
be the natural open immersions
We have an exact seq
$0 \to \mathcal{G} \to \bigoplus_j (i_j)_*(\mathcal{G}_{|D(f_j)}) \to \bigoplus (i_{j_1 j_2})_*(\mathcal{G}_{|D(f_{j_1} f_{j_2})})$
$\cong \widetilde{\Gamma(D(f_j), \mathcal{G})}$ on X $\qquad \cong \widetilde{\Gamma(D(f_{j_1} f_{j_2}), \mathcal{G})}$ on X
$(s_j) \longmapsto ((s_{j_1} - s_{j_2}))$
$\mathcal{G}$ being the kernel, is also QCoh.

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Thm: Let X be a scheme, $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$.
(i) $\mathcal{G}$ is QCoh iff for every affine open $U \subseteq X$, $\mathcal{G}_{|U} \xrightarrow{\sim} \tilde{M}_U$ for some $M_U \in \mathrm{Mod}_{\mathcal{O}_X(U)}$.
(ii) $\mathcal{G}$ is QCoh iff $\exists$ an affine open covering $X = \bigcup U_\lambda$ s.t. $\mathcal{G}_{|U_\lambda} \xrightarrow{\sim} \tilde{M}_\lambda$ for some $M_\lambda \in \mathrm{Mod}_{\mathcal{O}_X(U_\lambda)}$
(iii) When X is affine checking $\mathcal{G} \cong \tilde{M}$ iff $\mathcal{G}_{|U_\lambda} \cong \tilde{M}_\lambda$ for some affine cover $X = \bigcup U_\lambda$ and $M_\lambda \in \mathrm{Mod}_{\mathcal{O}_X(U_\lambda)}$.

Thm. X be a scheme, the category of QCoh sheaves $\mathrm{QCoh}(X)$ is abelian, admits arbitrary direct sum.

Pf HW:

§ Coherent Sheaves: (abv. coh)

The notion of coh sheaves on a scheme X is defined under the additional assumption X is locally noetherian (be careful, Hart assumes additionally X is noeth, which we do not).

Def. Let X be a locally noetherian scheme. $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}_X}$. $\mathcal{G}$ is called coherent (ab coh) if one of the following equivalent condn are satisfied (eq. is HW)
(i) $\mathcal{G}$ is QCoh and for every affine open $U \subseteq X$, $\Gamma(U, \mathcal{G})$ is a finitely generated $\mathcal{O}_X(U)$ mod.
(ii) There is an affine open covering $X = \bigcup_\lambda U_\lambda$ s.t. $\mathcal{G}_{|U_\lambda}$ is QCoh and $\Gamma(U_\lambda, \mathcal{G})$ is a f.g $\mathcal{O}_X(U_\lambda)$ mod for $\forall \lambda$.
(iii) For every affine open $U \subseteq X$ or on each element U_λ of an affine open covering $X = \bigcup U_\lambda$, $\mathcal{G}_{|U}$ or $\mathcal{G}_{|U_\lambda}$ is isom to $\tilde{M}$ for some f.g $\mathcal{O}_X(U)$ or $\mathcal{O}_X(U_\lambda)$-mod M.

Thm. $\mathrm{QCoh}(X)$ is closed under taking ker, coker, $\oplus$, $\otimes$, $\mathrm{QCoh}(X)$ is an abelian category.

Notation. $\mathrm{Coh}(X)$ = set of all coh $\mathcal{O}_X$-mods on X.

Thm. X, Y be noeth schemes.
i) $\varphi: \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin map between coh sheaves, $\ker \varphi$, $\operatorname{coker} \varphi$ coh.
(ii) $f: Y \to X$, $\mathcal{F} \in \mathrm{Coh}(X)$, $f^*\mathcal{F} \in \mathrm{Coh}(Y)$.
(iii) If f is finite $\mathcal{G} \in \mathrm{Coh}(Y)$, $f_*\mathcal{G} \in \mathrm{Coh}(X)$.
(iv) $\psi: \mathcal{F} \to \mathcal{G}$ $\mathcal{O}_X$-lin $\mathcal{G} \in \mathrm{QCoh}(X)$, $\mathcal{F} \in \mathrm{Coh}(X)$. $\operatorname{Im} \psi \in \mathrm{Coh}(X)$.
(v) $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$, so is $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$.

Thm. $f: X \to Y$ map of schemes, f quasi-compact and separated
(i.e inverse image of any quasi-compact subset is quasi compact or for some affine open cover $Y = \bigcup U_\lambda$, $f^{-1}(U_\lambda)$ is quasi-compact}; when X is noetherian, any f is automatically q. compact).
$\mathcal{F} \in \mathrm{QCoh}(X)$, $f_*\mathcal{F} \in \mathrm{QCoh}(Y)$.

Pf. W.L.O.G Y is affine. Choose a finite affine covering $f^{-1}(Y) = \bigcup_{j=1}^r U_j$
As X is separated, $U_{j_1} \cap U_{j_2}$ is affine
We have an exact seq
$0 \to \mathcal{F} \to \bigoplus (i_j)_*(\mathcal{F}_{|U_j}) \to \bigoplus_{j_1, j_2} (i_{j_1 j_2})_*(\mathcal{F}_{|U_{j_1} \cap U_{j_2}})$
$(s_j) \longmapsto (s_{j_1} - s_{j_2})$

Applying f_* get an exact seq

$0 \to f_*\mathcal{F} \to \bigoplus f_*(i_j)_*(\mathcal{F}_{|U_j}) \to \bigoplus f_*(i_{j_1 j_2})_*\mathcal{F}_{|U_{j_1} \cap U_{j_2}}$
$f_*(i_j)_* = (f \cdot i_j)_*$ $\quad f \cdot i_j: U_j \to Y$, both U_j, Y are affine, so the 2nd and 3rd terms are QCoh. So the kernel is QCoh.